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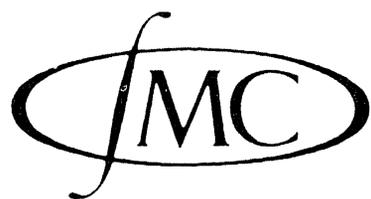
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Some analytical aspects of the
Tricomi Problem

by

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"Some analytical aspects of the Tricomi Problem"

Introduction

In this paper we study some of the analytical aspects of equations of the mixed type. Such equations are frequently encountered in the theory of transonic flow and then give rise to a special boundary value problem, called the Tricomi-problem. The analytical treatment of such problems may be divided in three simpler problems, which are in essence

- 1^o. A Dirichlet problem.
- 2^o. A Cauchy problem.
- 3^o. The problem of solving a singular integral equation.

The first problem will hardly be discussed. For the Cauchy problem we presented a very simple method leading to the solution of a class of equations including the Tricomi and Gellerstedt equations. In connection with the third problem we studied uniqueness proofs for equations of the mixed type. According to the Fredholm alternative for integral equations, this enables us to decide for which problems there really exist unique solutions, a very important problem in transonic flow. To obtain explicit solutions for the Tricomi problem it seems desirable to employ numerical methods. In a forthcoming paper we discuss this problem.

1. Equations of the mixed type

We consider a linear partial differential equation of the second order

$$(1.1) \quad L\psi = A\psi_{xx} + 2B\psi_{xy} + C\psi_{yy} + D\psi_x + E\psi_y + F\psi = 0,$$

with real coefficients which are functions of the two independent variables x, y defined in a domain \mathcal{R} of the x, y plane. If the discriminant $\Delta = B^2 - AC$ changes sign in \mathcal{R} (1.1) is called an equation of the mixed type. According to the sign of Δ the domain \mathcal{R} may be divided in elliptical, parabolical and hyperbolical points.

According to the standard theory to (1.1) we may associate characteristic variables ζ and η such that $\zeta(x,y) = \text{constant}$ and $\eta(x,y) = \text{constant}$ are solutions of

$$(1.2) \quad \frac{dy}{dx} = \frac{-B \pm \sqrt{\Delta}}{A}.$$

These variables are either real or complex conjugate.

In terms of the new independent variables ζ and η equation (1.1) may be written as

$$(1.1') \quad \psi_{\zeta\eta} + D_1\psi_{\zeta} + E_1\psi_{\eta} + F_1\psi = 0.$$

The variables ζ , η may be replaced by the following variables which are always real

$$(1.3) \quad \theta = \frac{1}{2} (\zeta + \eta) \quad \sigma = \left(\frac{3}{4}\right)^{2/3} \{-(\zeta - \eta)^2\}^{1/3},$$

and transformation to the variables θ and σ yields the equation

$$(1.1'') \quad \sigma\psi_{\theta\theta} + \psi_{\sigma\sigma} + d\psi_{\theta} + e\psi_{\sigma} + f\psi = 0.$$

In analogy to the elliptic and hyperbolic equations this may be called the normal form of the equations of the mixed type. The equation

$$(1.4) \quad \sigma\psi_{\theta\theta} + \psi_{\sigma\sigma} = 0$$

then corresponds to the equation of Laplace and the wave equation. Equation (1.4) is called the Tricomi equation and may be considered as a natural completion of the usual classification of partial differential equations.

2. Gasdynamics

Equations of mixed type appear in the theory of gasdynamics when the streamfunction in the hodograph-plane is calculated.

We consider plane stationary potential flow which in the physical (x,y) -plane can be described by

$$(2.1) \quad c^2 \nabla \rho + \rho (\vec{q} \cdot \nabla) \vec{q} = 0,$$

$$(2.2) \quad \nabla \cdot \rho \vec{q} = 0,$$

$$(2.3) \quad \nabla \times \vec{q} = 0,$$

where ρ is the density and \vec{q} is the velocity. Further $c^2 = \frac{\partial p}{\partial \rho}$ follows from the known equation of state $p = f(\rho)$. We may introduce a stream function ψ and a potential function ϕ for which

$$\nabla \psi = \vec{r} = (-\rho v, \rho u), \quad \nabla \phi = \vec{q} = (u, v).$$

The streamlines $\psi = \text{constant}$ and the potential lines $\phi = \text{constant}$ define

a curvilinear orthogonal coordinate system in which the equations (2.2) and (2.3) may easily be transformed by means of the formula [28]

$$\nabla \cdot \vec{a} = \frac{\partial}{(h \partial \zeta)_i} a_i - k_i a_i,$$

which describes the transition of the divergence in the (x, y) -plane to a generalized (ζ_i) -system with scalefactors h_i and coordinate curvatures k_i . In this case we have $\zeta_1 = \phi$ and $\zeta_2 = \psi$, so that

$$(2.4) \quad h_1 = \frac{1}{q}, \quad h_2 = \frac{1}{\rho q}.$$

If θ is the angle formed by the vector \vec{q} and the positive x direction we obtain

$$(2.5) \quad k_1 = -\rho q \frac{\partial}{\partial \psi} \theta, \quad k_2 = q \frac{\partial}{\partial \phi} \theta.$$

Since in the (ϕ, ψ) -plane we have $\rho \vec{q} = (\rho q, 0)$ and $\frac{1}{\rho} \vec{r} = (0, q)$ the equations (2.2) and (2.3) are transformed into

$$(2.2') \quad q(\rho q)_\phi + (\rho q)^2 \theta_\psi = 0,$$

$$(2.3') \quad \rho q q_\psi - q^2 \theta_\phi = 0.$$

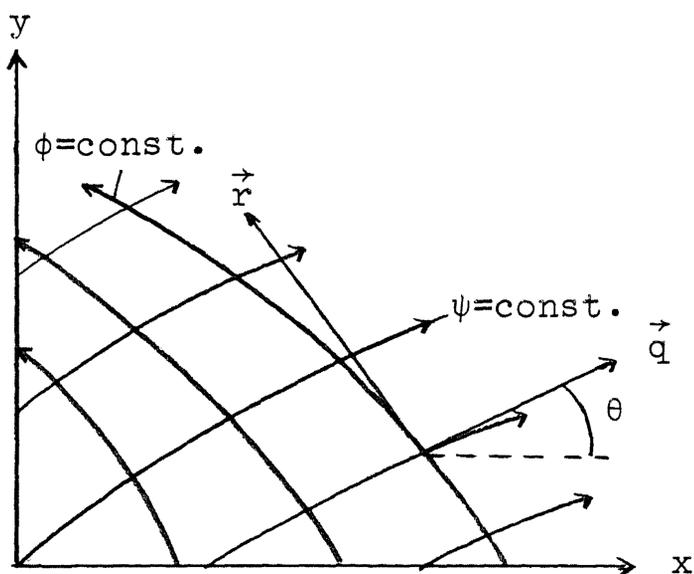


fig. 1

It is our purpose to take the hodograph variables q and θ as new independent variables. Using the relations $\frac{\partial}{\partial \phi} = q_{\phi} \frac{\partial}{\partial q} + \theta_{\phi} \frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial \psi} = q_{\psi} \frac{\partial}{\partial q} + \theta_{\psi} \frac{\partial}{\partial \theta}$ we get

$$1 = q_{\phi} \phi_q + \theta_{\phi} \phi_{\theta}, \quad 0 = q_{\phi} \psi_q + \theta_{\phi} \psi_{\theta},$$

and

$$1 = q_{\psi} \psi_q + \theta_{\psi} \psi_{\theta}, \quad 0 = q_{\psi} \phi_q + \theta_{\psi} \phi_{\theta}.$$

If the transformation determinant D does not vanish we have

$$q_{\phi} = \frac{\psi_{\theta}}{D}, \quad \theta_{\phi} = -\frac{\psi_q}{D}, \quad q_{\psi} = \frac{\phi_{\theta}}{D}, \quad \theta_{\psi} = \frac{\phi_q}{D}.$$

If we transform the equations (2.2') and (2.3') to the hodograph plane the term $q(\rho q)_{\phi}$ gives rise to a nonlinearity, for we can write

$$q(\rho q)_{\phi} = q\rho q_{\phi} + q^2\rho_{\phi},$$

where $q^2\rho_{\phi}$ causes the nonlinearity. We eliminate $q^2\rho_{\phi}$ with the aid of equation (2.1). The component of $\nabla\rho$ along a streamline is given by

$$q \frac{\partial}{\partial \phi} \rho = -\frac{\rho}{c^2} q^2 \frac{\partial}{\partial \phi} q = -\frac{\rho q^2}{c^2} \frac{1}{D} \psi_{\theta}.$$

The following linear system is obtained

$$(2.6) \quad \begin{cases} q\psi_q + \rho\phi_{\theta} = 0 \\ \rho q\phi_q + (1 - M^2)\psi_{\theta} = 0, \end{cases}$$

where $M = q/c$ is the local Mach number.

We suppose that ρ and M depend only on q . Then by differentiation with respect to q and θ , ϕ can be eliminated from (2.6) and we obtain an equation of the type (1.1)

$$(2.7) \quad \frac{\partial}{\partial q} \left(\frac{q}{\rho} \psi_q \right) - \frac{M^2 - 1}{\rho q} \psi_{\theta\theta} = 0.$$

The discriminant of this equation is given by

$$(2.8) \quad \Delta = \frac{M^2 - 1}{q^2}$$

which changes of sign for $M = 1$, i.e. when q equals the local velocity of sound. Hence in the points where the flow is changing from subsonic to supersonic equation (2.7) changes of type. In other words equation (2.7) is of the mixed type and the sonic line corresponds with the parabolic line.

Next we investigate the physical meaning of the characteristics of equation (2.7). Let l^\pm be the characteristic lines in the (x,y) -plane with the unit tangent \vec{l} . Along l we have

$$(2.9) \quad \frac{dq}{d\theta} = \frac{\frac{\partial q}{\partial l}}{\frac{\partial \theta}{\partial l}} = \frac{\vec{l} \cdot \nabla q}{\vec{l} \cdot \nabla \theta} = \frac{(l_1, l_2) \cdot (q q_\phi, \rho q q_\psi)}{(l_1, l_2) \cdot (q \theta_\phi, \rho q \theta_\psi)},$$

where l_1 and l_2 are the components of \vec{l} along the potential lines and the streamlines respectively.

From (2.6) we have

$$(2.10) \quad \nabla \theta = (\rho q_\psi, (M^2 - 1)q_\phi).$$

Substituting this into (2.9) and recalling that $l_1 = \pm \sqrt{1 - l_2^2}$, we obtain

$$(2.11) \quad \frac{dq}{d\theta} = \frac{\pm q}{\sqrt{M^2 - 1}} \cdot \frac{\pm \rho l_2 q_\psi + \sqrt{1 - l_2^2} q_\phi}{\frac{\pm \rho \sqrt{1 - l_2^2} q_\psi}{\sqrt{M^2 - 1}} + l_2 \sqrt{M^2 - 1} q_\phi}.$$

On the other hand the line l corresponds to a characteristic of (2.7) so that

$$(2.12) \quad \frac{dq}{d\theta} = \frac{\pm q}{\sqrt{M^2 - 1}}.$$

From (2.11) and (2.12) we obtain the result

$$l_2 = \pm \frac{1}{M}.$$

Let α be defined as the angle between \vec{l} and \vec{q} (see figure 2), then

$$\alpha = \arcsin\left(\pm \frac{1}{M}\right).$$

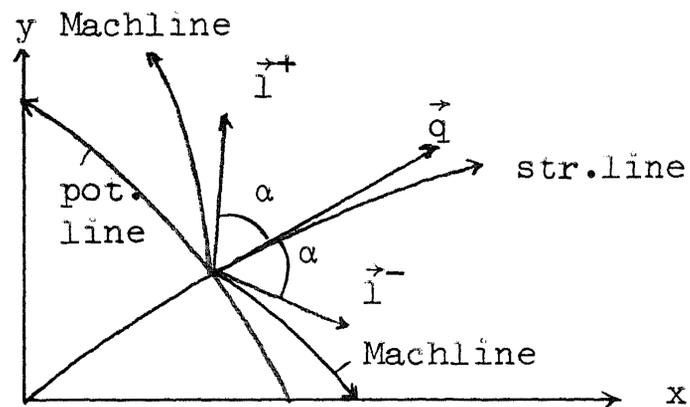


fig. 2

This means that the lines l^{\pm} are identical to the so-called Mach lines, thus the Mach lines correspond to the characteristics.

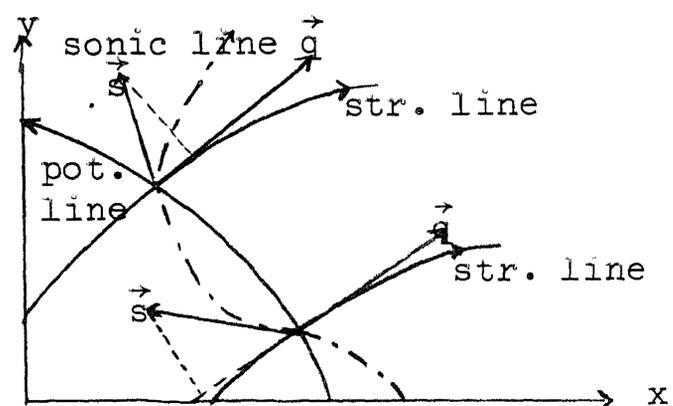


fig. 3

One can also state something about the orientation of the sonic line in the physical plane (figure 3). Suppose that the flow at the "left" hand side of the sonic line s is subsonic and let \vec{s} be the unit tangent along s then from (2.10) we have

$$(2.13) \quad \frac{\partial \theta}{\partial s} = \vec{s} \cdot \nabla \theta = \rho s_1 q_\psi.$$

In figure 3 two different situations are illustrated, showing that $\frac{\partial \theta}{\partial s}$ is always negative. In general we have a monotonic behaviour of the variable θ along the sonic line.

In some important cases it is possible to give an explicit formulation of the boundary value problem in the hodograph plane by means of this method. In particular the transonic flow through a two-dimensional pipe (see figure 4) can be treated in this way.

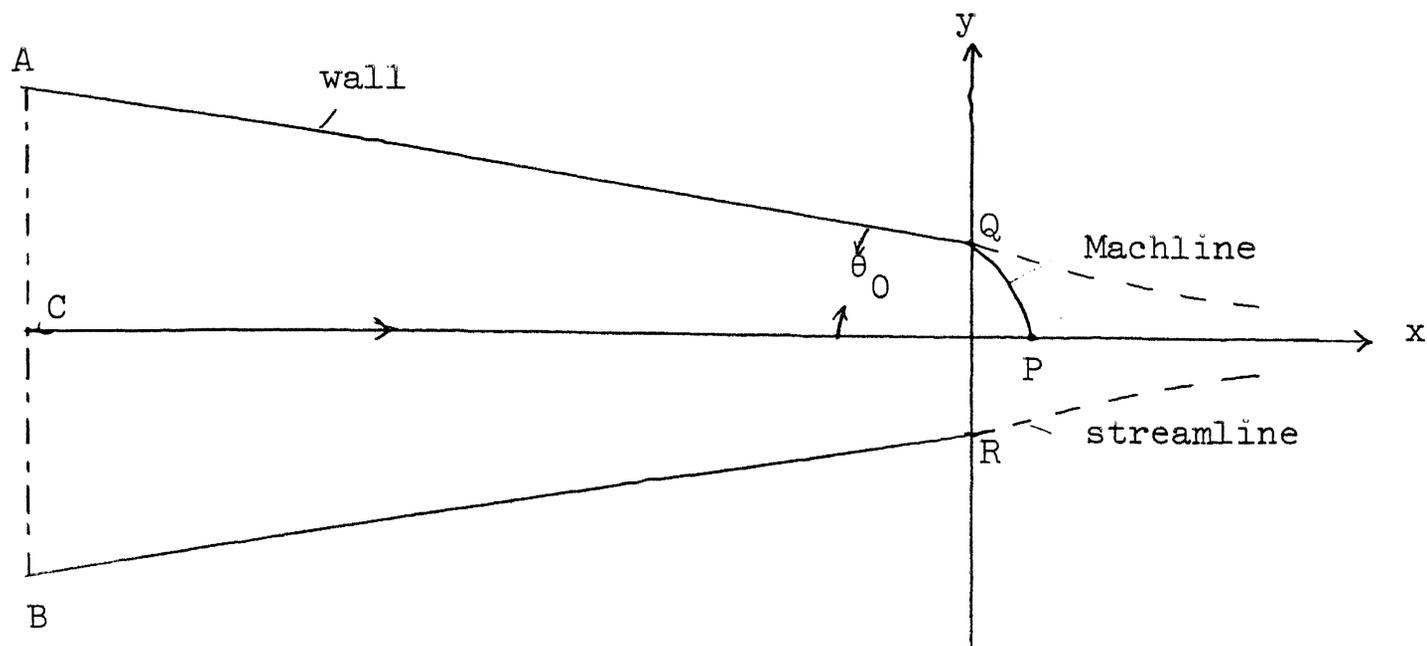


fig. 4

We suppose that the flow is symmetrical and that the walls and the axis of symmetry (x-axis) are streamlines. According to Frankl' ([4], [29]) the point Q in figure 4 will be a sonic point and the Mach line starting in the sonic point P on the x-axis will run through Q. These data are sufficient to give a formulation of the boundary value problem in the hodograph plane. In the subsonic region the boundary is formed by the lines $\theta = \pm \theta_0$ (figure 4) and a line $q(\theta)$ corresponding to the velocity distribution at the entrance of the pipe. In the supersonic region the boundaries are formed by the Mach lines, i.e. the characteristics of the equation. In the hodograph plane the point Q lies on a characteristic.

Instead of the variables q and θ we use variables σ and θ by means of which equation (2.7) can be transformed into a more appropriate form.

$$(2.14) \quad K(\sigma)\psi_{\theta\theta} + \psi_{\sigma\sigma} = 0,$$

where σ is defined by $\sigma = \int_q^{q_{cr}} \frac{\rho}{q} dq$ and $K(\sigma) = \frac{1 - M^2}{\rho^2}$.

The point $q = 0$ corresponds to $\sigma = \infty$ and the sonic line corresponds to the θ -axis. In the new hodograph plane the geometry of the boundary value problem may be sketched as in figure 5.

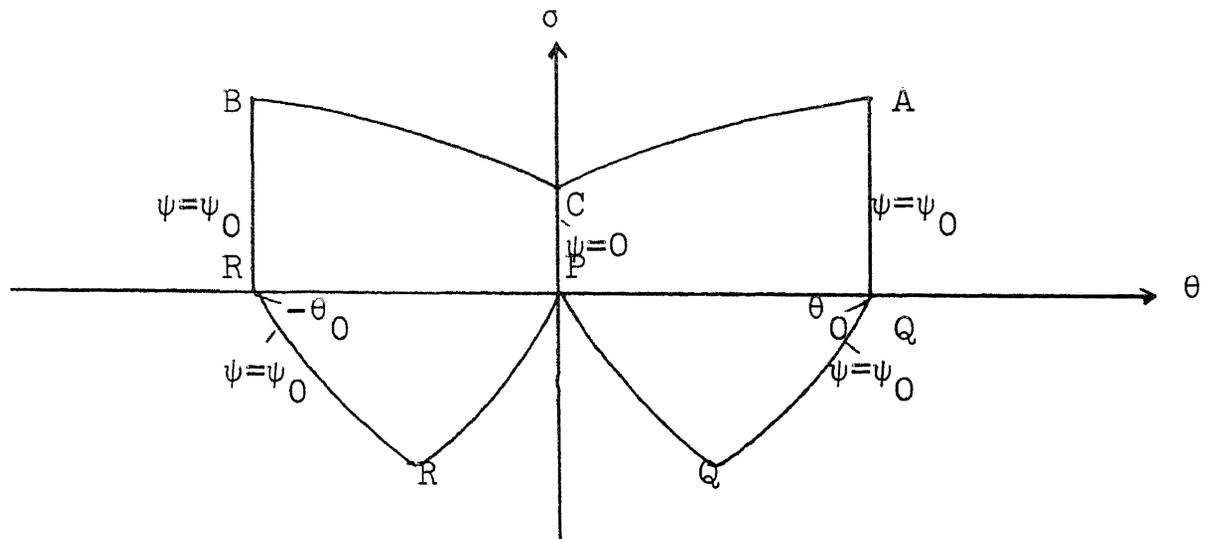


fig. 5

Since the flow is symmetrical with respect to the x -axis the boundary value problem is symmetrical with respect to the line $\theta = 0$. Therefore it is sufficient to consider only one half of the region say that of $\theta \geq 0$.

The curve CA corresponds to the velocity distribution at the entrance of the pipe. It is necessary to prescribe this distribution (section 6) beforehand. Equation (2.14) is called the Chaplygin equation and the boundary value problem can be considered as a special case of the Tricomi problem. Before this problem appeared in the theory of gasdynamics, Tricomi (1923) considered a slightly more general boundary value problem, in which the elliptic boundary was an arbitrary Jordan-curve. He restricted the considerations to the Tricomi equation (1.4). In this connection we remark that the Chaplygin equation resembles the Tricomi equation very closely if σ is small and the fluid is an ideal adiabatic gas. In that case we have

$$(2.15) \quad M^2 = 2\beta \frac{q^2}{2\beta - q^2}, \quad \rho = \rho_0 \left(\frac{2\beta - q^2}{2\beta} \right)^\beta,$$

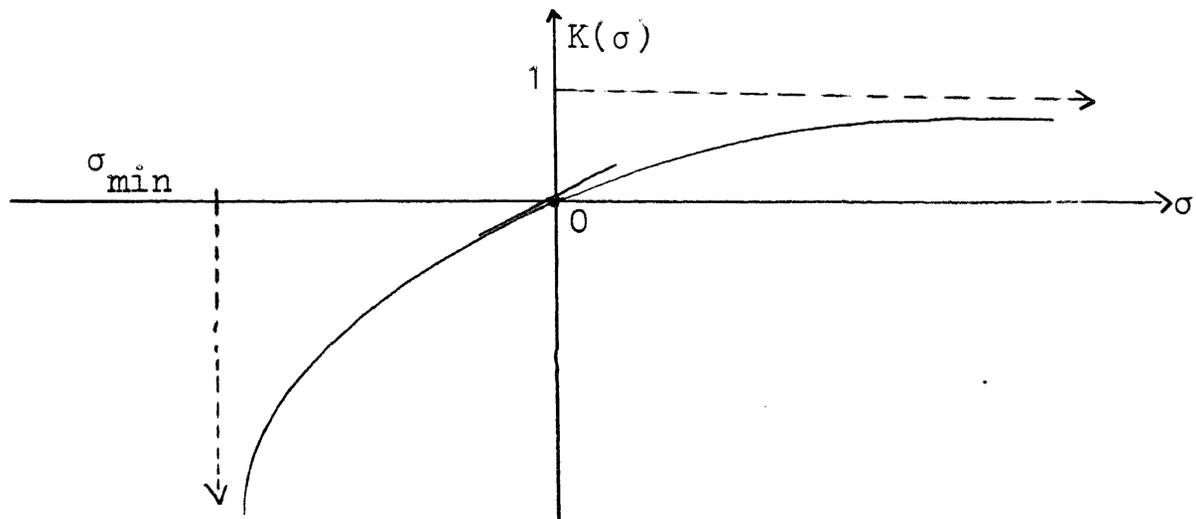
where ρ_0 and β are constants.

Choosing $\rho_0 = 1$ we find from (2.14)

$$K(0) = 0, \quad K'(\sigma) > 0, \quad K'(0) = 1$$

(2.16) and

$$\lim_{\sigma \rightarrow \infty} K(\sigma) = 1, \quad \lim_{\sigma \rightarrow \sigma(\sqrt{2\beta})} K(\sigma) = -\infty.$$



We conclude this section by mentioning a remarkable transformation between the Chaplygin equation and the Tricomi equation. Germain and Liger [20] proved that it is possible to reduce the Chaplygin equation (2.14) to the Tricomi equation by means of the transformations

$$(2.17) \quad 3y = 2z^{3/2}, \quad x+iy = \operatorname{tg}\left(\frac{\theta}{a} + is\right), \quad \beta(s)\psi(\theta,s) = y^{1/6}u(z,x),$$

where $a = \text{constant}$.

This yields

$$zu_{xx} + u_{zz} = 0.$$

This confirms the central place of the Tricomi equation in gasdynamics.

3. Survey of analytical methods

In this section we give an outline of some methods to treat boundary value problems such as the Tricomi problem. We consider a mixed equation in an elliptic domain \mathcal{R}^+ , a parabolic line C and a

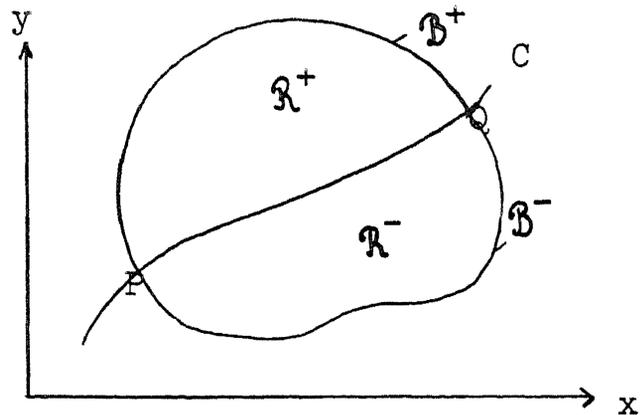


fig. 6

hyperbolic domain \mathcal{R}^- (see figure 6).

Since the behaviour of the solution ψ of the equation $L\psi = 0$ is different in the elliptic domain from that in the hyperbolic domain, one attempts to solve the following problems separately

- (E) The explicit solution of the equation in the elliptic domain in terms of the values ψ , ψ_x or ψ_y on the parabolic line, which satisfies the conditions along the boundary \mathcal{B}^+ .
- (H) The same problem for the equation in the hyperbolic domain.
- (P) The matching of the two solutions on the parabolic line by elimination of the used values for ψ , ψ_x or ψ_y on the parabolic line.

The elliptic problem (E) - Until this time the greater part of the theory concerns equations of the type

$$(3.1) \quad K(y)\psi_{xx} + \psi_{yy} = 0,$$

with boundary conditions of the first kind along the curve \mathcal{B}^+ (figure 6).

Where $K(y) \equiv y$ (equation of Tricomi) the problem may be reduced to a potential problem under the transformations

$$r = \frac{2}{3} y^{3/2}, \quad z = x,$$

yielding

$$(3.2) \quad \psi_{zz} + \psi_{rr} + \frac{1}{3r} \psi_r = 0.$$

The solution of (3.2) may be considered as an axial-symmetric solution of the Laplace equation in a $2\frac{1}{3}$ dimensional region.

Holmgren (1916) was the first to give explicitly Green's function of the problem [22]. Later Weinstein (1948), applying his generalized axial-symmetric potential theory (the so-called GASP-method) [37, 38, 39, 40, 41], and Germain and Bader (1950), with a group of transformations [7, 13, 14, 15], constructed other fundamental solutions for (3.2).

For the more general equation of Chaplygin, Germain (1952) gives an integral representation of a fundamental solution by applying the generalized Fourier-theory of Laurent Schwarz [11].

With the aid of the fundamental solution of Holmgren, Tricomi [36], Gellerstedt [6] and Protter [32], give the solution ψ in terms of $v(x) = \psi_y(x,0)$ for the cases $K(y) \equiv y$ (Tricomi equation), $K(y) \equiv y^m$ (Gellerstedt equation) and $K(y) \in C^3$ (Chaplygin equation) respectively.

The hyperbolic problem (H) - We considered only methods for the Chaplygin equation yielding ψ in terms of $v(x)$ and where ψ satisfied the boundary conditions on \mathcal{B} . To solve this problem one considers first the Cauchy problem on the parabolic line.

Tricomi and Gellerstedt solved this problem in a very complicated manner (due to Hadamard) for the cases $K(y) \equiv y$ and $K(y) \equiv y^m$ respectively. Protter generalized the method for $K(y) \in C^3$ [32].

There are a large number of other methods to solve this Cauchy problem. In the first place the method of Riemann [12, 13, 19], but also the GASP-method of Weinstein, the transformation-groups of Germain and Bader, and the integral-operators of Bergman [3] may be used to construct fundamental solutions with the singularity in the hyperbolic region. One may give directly an integral-representation for the solution of equations of the Gellerstedt type (Bergman [2]), when we approximate the initial-functions by polynomials. In the following section we derive by very elementary methods also an integral-representation, for the solutions of a class of equations, including the Gellerstedt type. After we have found ψ in terms of $v(x)$ and $\tau(x) = \psi(x,0)$, we substitute the boundary conditions along \mathcal{B}^- , obtaining a functional relation between $\tau(x)$ and $v(x)$.

The elimination problem (P) - From the solution of problem (E) we have an expression for $\tau(x)$ in terms of $v(x)$ and substituting this into the relation obtained by solving problem (H) we get a functional relation for $v(x)$ alone. If the conditions on \mathcal{B}^- are of the first kind this relation will be a singular integral equation for $v(x)$. For example, for the Tricomi equation and the Tricomi boundary value problem we get [19]

$$(3.3) \quad v(x) + \frac{1}{\pi\sqrt{3}} \int_0^1 \left(\frac{t}{x}\right)^{2/3} \left(\frac{1}{t-x} - \frac{2}{t+x-2tx}\right)v(t)dt + \\ + \int_0^1 G(x,t)v(t)dt = F(x),$$

where $G(x,t)$ and $F(x)$ are given functions.

Without solving this integral equation we can say something about the uniqueness and the existence of the solution. For the Chaplygin equation and the Tricomi problem, the Fredholm-alternative applies to the resulting integral equation, i.e., either there exists a unique solution for the boundary value problem, or there exist non-trivial solutions of the equation $h\psi = 0$ satisfying homogeneous boundary conditions. An uniqueness-theorem proves the unique existence of a solution, as then the kernel of h is trivial and the second possibility is contradicted.

4. The Cauchy problem on the parabolic line

First we consider the solution of the wave equation

$$(4.1) \quad \psi_{xx} - \psi_{yy} = 0.$$

It is well-known that the solution of the Cauchy problem on the x -axis is given by

$$(4.2) \quad \psi(x,y) = \left[\frac{1}{2}\tau(x-y) + \frac{1}{2}\tau(x+y) \right] + \left[y \int_0^1 v(x+(1-2t)y)dt \right],$$

where $\tau(x) = \psi(x,0)$ and $v(x) = \psi_y(x,0)$ on the interval $[0,1]$.

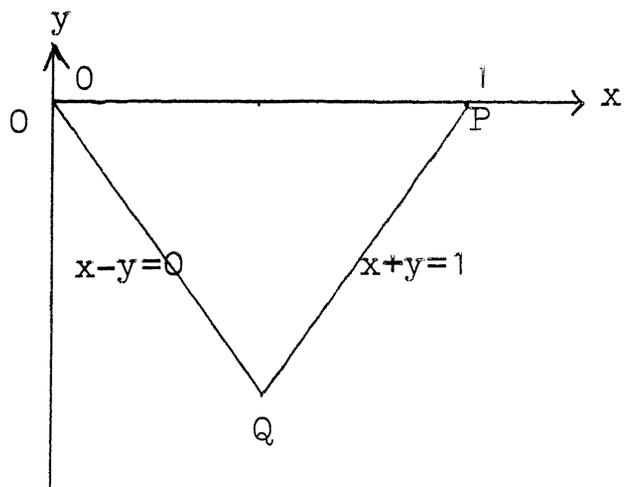


fig. 7

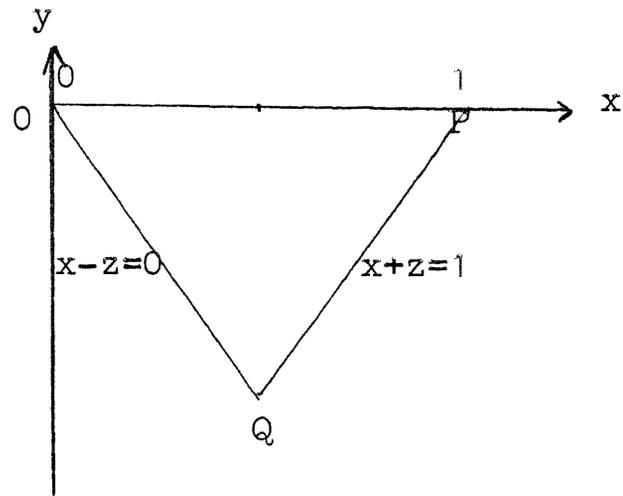


fig. 8

The functions $\tau(x)$ and $\nu(x)$ completely define the solution in the characteristic triangle OPQ . We may interpret the solution $\psi(x,y)$ as a weighted sum of τ and ν over the interval of dependence $[x-y, x+y]$ of the point (x,y) .

We consider now the Chaplygin equation

$$(4.3) \quad L\psi \equiv K(y)\psi_{xx} + \psi_{yy} = 0.$$

Define

$$(4.4) \quad z(y) = \int_0^y \sqrt{-K(y)} dy,$$

then the characteristics are given by

$$(4.5) \quad x \pm z = \text{constant},$$

and the interval of dependence on the parabolic line of the point (x,y) is given by

$$(4.6) \quad [x - z(y), x + z(y)].$$

The solution (4.2) suggests the following integral representation for the solution of the Chaplygin equation

$$(4.7) \quad \psi(x,y) = \int_0^1 F(x + (1-2t)z)v(t)dt + y \int_0^1 G(x+(1-2t)z)w(t)dt,$$

in which we assume F to depend on τ and G on ν .

From $L\psi \equiv 0$ we must deduce the functions F , G , v and w .

We divide the problem into two more simple problems with $v(x) \equiv 0$ and $\tau(x) \equiv 0$ respectively. Superposition of the two solutions gives the solution of the general Cauchy problem.

a) $v(x) \equiv 0$ - We write $f(x,z,t) = x + (1-2t)z$ and applying the operator L to F , we obtain

$$(4.8) \quad \begin{aligned} LF &= (Kf_{xx} + f_{yy})F'(f) + (Kf_x^2 + f_y^2)F''(f) = \\ &= (1-2t)z''F'(f) + (K + (1-2t)^2z'^2)F''(f). \end{aligned}$$

It is our intention to transform LF into MF , where M is an operator only in the variable t .

From the identities

$$\frac{\partial}{\partial t} F = f_t F'(f) = -2zF'(f)$$

and

$$\frac{\partial^2}{\partial t^2} F = f_{tt} F'(f) + f_t^2 F''(f) = 4z^2 F''(f),$$

we see that the following relations have to be satisfied

$$(4.9) \quad (1-2t)z'' = -2z a(t) p(y),$$

$$(4.10) \quad K + (1-2t)^2 z'^2 = 4z^2 b(t) p(y),$$

where $a(t)$ and $b(t)$ are certain functions of t and $p(y)$ is a certain function of y .

We see immediately that

$$(4.11) \quad a(t) = \lambda(1-2t) \quad \text{and} \quad b(t) = t(1-t),$$

where λ is a constant, while $z = z(y)$ must satisfy the equations

$$\begin{aligned} z'' + 2\lambda p(y)z &= 0 \\ z'^2 + p(y)z^2 &= 0. \end{aligned}$$

If we eliminate $p(y)$ from these relations we find a non-linear equation for $z(y)$

$$(4.12) \quad zz'' - 2\lambda z'^2 = 0.$$

This relation restricts the class of equations to which the method applies. For the moment we assume that (4.12) is satisfied.

Next we substitute (4.9) and (4.10) into (4.8), obtaining

$$(4.13) \quad L\psi_1 = p(y) \int_0^1 (a(t) \frac{\partial}{\partial t} + b(t) \frac{\partial^2}{\partial t^2}) F \cdot v(t) dt.$$

We define the operator M :

$$(4.14) \quad M = a(t) \frac{\partial}{\partial t} + b(t) \frac{\partial^2}{\partial t^2},$$

with the property

$$(4.15) \quad v(t)MF = FM^*v(t) + \frac{\partial}{\partial t} P(v,F),$$

where

$$(4.16) \quad M^* = \frac{\partial^2}{\partial t^2} b(t) - \frac{\partial}{\partial t} a(t)$$

and

$$(4.17) \quad P(v,F) = bvF_t - F(bv)_t + avF.$$

Substituting (4.15) into (4.13) gives:

$$(4.18) \quad L\psi_1 = p(y) \int_0^1 FM^*v dt + p(y) \int_0^1 \frac{\partial}{\partial t} P(v,F) dt.$$

$L\psi_1$ becomes equal to zero if we set $M^*v = 0$ and $P(v,F) \Big|_{t=0}^{t=1} = 0$.
In so doing we obtain,

$$(vb)_t - av = c = \text{constant},$$

$$vbF_t \Big|_{t=0}^{t=1} = c.$$

Choosing $c = 0$ this results in

$$(4.19) \quad v(t) = C_1 t^{\lambda-1} (1-t)^{\lambda-1},$$

$$(4.20) \quad \left[t^\lambda (1-t)^\lambda F'(x+(1-2t)z) \right]_{t=0}^{t=1} = 0 \text{ on } \mathfrak{R}^-,$$

where C_1 is a constant.

From (4.20) we see that $\lambda > 0$.

Now that we have found $v(t)$, the solution $\psi_1(x,y)$ becomes

$$\psi_1(x,y) = C_1 \int_0^1 F(x + (1-2t)z) t^{\lambda-1} (1-t)^{\lambda-1} dt,$$

and substituting $\psi(x,0) = \tau(x)$ we find that

$$\tau(x) = C_1 F(x) \int_0^1 t^{\lambda-1} (1-t)^{\lambda-1} dt = C_1 B(\lambda, \lambda) F(x),$$

where $B(\lambda, \lambda)$ is the Beta-function.

Choosing $C_1 = B^{-1}(\lambda, \lambda)$ and $F(x) = \tau(x)$, we finally have

$$(4.21) \quad \psi_1(x,y) = B^{-1}(\lambda, \lambda) \int_0^1 t^{(\lambda-1)} (1-t)^{1-\lambda} \tau(x+(1-2t)z) dt,$$

which satisfies the Chaplygin equation and the boundary condition $\psi(x,0) = \tau(x)$, $\psi_y(x,0) = 0$, when $z(y)$ is a solution of (4.12).

b) $\tau(x) = 0$ - The solution of this second Cauchy problem follows exactly the method used in case a).

One finds that $L\psi_2$ may be written as

$$(4.22) \quad L\psi_2 = q(y) \int_0^1 \left(c(t) \frac{\partial}{\partial t} + d(t) \frac{\partial^2}{\partial t^2} \right) G \cdot w(t) dt,$$

where $q(y)$, $z(y)$, $c(t)$ and $d(t)$ must satisfy the equations

$$(1-2t)(yz'' + 2z') = -2z q(y) c(t),$$

$$t(t-1)yz'^2 = z^2 d(t) q(y).$$

These equations are solved by the relations:

$$\begin{aligned} c(t) &= \mu(1-2t), \\ d(t) &= t(1-t), \\ (4.23) \quad y(z''z - 2\mu z'^2) + 2zz' &= 0, \end{aligned}$$

where μ is a constant.

Again we define an operator $M = c \frac{\partial}{\partial t} + d \frac{\partial^2}{\partial t^2}$. If M^* is the adjoint operator, we have

$$w(t)MG = GM^*w(t) + \frac{\partial}{\partial t} P(w,G),$$

where $P(w,G)$ is the corresponding bilinear concomitant.

Putting this into (4.22) we find that $L\psi_2 = 0$, if

$$(4.24) \quad w(t) = C_2 t^{\mu-1} (1-t)^{\mu-1},$$

$$(4.25) \quad \left[t^\mu (1-t)^\mu G'(x + (1-2t)z) \right]_{t=0}^{t=1} = 0,$$

where C_2 is a constant.

Condition (4.25) implies that $\mu > 0$. With (4.24) and remembering that $\psi_y(x,0) = v(x)$, the solution $\psi_2(x,y)$ becomes

$$(4.27) \quad \psi_2(x,y) = B^{-1}(\mu,\mu)y \int_0^1 t^{\mu-1} (1-t)^{\mu-1} v(x+(1-2t)z) dt.$$

c) The general Cauchy problem - Obviously the solution of the general Cauchy problem is given by

$$(4.28) \quad \psi = \psi_1 + \psi_2.$$

We have seen that the following conditions must be satisfied

- 1) $\tau(x)$ and $v(x)$ differentiable.
- 2) $z(y) = \int_0^y \sqrt{-K(y)} dy$ satisfies the equations (4.12) and (4.23).

From (4.12) and (4.23) we deduce that

$$\lambda = \frac{zz''}{zz'^2} \quad \text{and} \quad \mu - \lambda = \frac{z}{yz'},$$

with the solution $z = \text{const}(-y)^{\frac{1}{\mu-\lambda}}$ and $\lambda + \mu = 1$.

For $K(y)$ we finally find

$$(4.29) \quad K(y) = -|\text{const}| (-y)^{\frac{2\lambda}{1-2\lambda}},$$

where $\lambda > 0$.

The special case of the Tricomi equation is obtained for $\lambda = 1/6$.

5. Uniqueness-proofs

We consider in the domain \mathcal{R} the equation

$$(5.1) \quad L\psi = f,$$

with boundary conditions $M\psi = 0$ along the boundary \mathcal{B} and we denote the sets $\{\psi \mid L\psi = 0\}$ and $\{\psi \mid M\psi = 0\}$ by \mathcal{N}_L and \mathcal{N}_M . Now equation (5.1) has an unique solution (if there is a solution) in \mathcal{N}_M when $\mathcal{N}_L \cap \mathcal{N}_M = \{0\}$. Therefore, if we can construct a functional which vanishes on \mathcal{N}_L , but which is definite on \mathcal{N}_M , we are sure that equation (5.1) has at most one solution in \mathcal{N}_M . In this way the methods, employed by Morawetz [25] and Protter [31, 32] for the Tricomi problem and related problems may be interpreted.

Morawetz considered the functional

$$(5.2) \quad F(\psi) = \int_{\mathcal{R}} (K(y)\psi_{xx} + \psi_{yy})(a(x,y)\psi_x + b(x,y)\psi_y) dx dy,$$

where $a(x,y)$ and $b(x,y)$ must be chosen in such a way that $F(\psi)$ is definite on \mathcal{N}_M . For boundary conditions of the first kind, Morawetz proved that one cannot find functions $a(x,y)$ and $b(x,y)$ for every unique solvable boundary value problem, which make $F(\psi)$ definite.

The elliptical boundary \mathcal{B}^+ must always satisfy certain minimal conditions.

Protter considered a slightly more general functional

$$(5.3) \quad F(\psi) = \int_{\mathcal{R}} (K(y)\psi_{xx} + \psi_{yy})(a\psi_x + b\psi_y + c) dx dy,$$

which enabled him to treat arbitrary elliptical boundaries when $K(y)$ satisfies certain conditions. In this section we give a matrix formulation of the methods of Morawetz and Protter, which is related to the matrix method of Friedrichs for symmetric positive linear differential equations.

Let \vec{u} be the vector $(\psi_x, \psi_y, \psi)^T$, then the Chaplygin equation may be written as

$$\begin{aligned} K(y)\partial_1 u_1 + \partial_2 u_2 &= g, \\ \partial_2 u_2 - \partial_1 u_2 &= 0, \\ u_1 - \partial_1 u_3 &= 0, \\ u_2 - \partial_2 u_3 &= 0, \end{aligned}$$

where $\partial_1 = \frac{\partial}{\partial x}$ and $\partial_2 = \frac{\partial}{\partial y}$.

The boundary conditions are represented by $M\vec{u} = 0$.

We define the matrices

$$(5.4) \quad (a_{kj}^1 \partial_1) = \begin{pmatrix} K\partial_1 & \partial_2 & 0 \\ \partial_2 & -\partial_1 & 0 \\ 0 & 0 & -\partial_1 \\ 0 & 0 & -\partial_2 \end{pmatrix}; \quad (c_{kj}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad (g_k) = \begin{pmatrix} g \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

In index-notation we have for the Chaplygin equation

$$(5.5) \quad (a_{kj}^1 \partial_1 + c_{kj})u_j = g_k.$$

We transform this equation by multiplying it by a 3×4 matrix (t_{ik}) , obtaining

$$(5.6) \quad (t_{ik} a_{kj}^1 \partial_1 + t_{ik} c_{kj})u_j = t_{ik} g_k,$$

or symbolically

$$(5.6') \quad L\vec{u} = \vec{f}.$$

The functional $F(\psi)$ is now defined by

$$(5.7) \quad F(\psi) = \int_{\mathcal{R}} L\vec{u} \cdot \vec{u} dx dy,$$

where the dot means the vector-product between $L\vec{u}$ and \vec{u} .

We see immediately that a non-trivial solution of the homogeneous Chaplygin equation in \mathcal{A}_M makes $L\vec{u}$ zero and therefore, $F(\psi)$ vanishes for an element $\psi \neq 0$ with $(\psi_x, \psi_y, \psi) \in \mathcal{A}_M$. The definiteness of $F(\psi)$ excludes this possibility and proves the uniqueness.

We will now investigate the functional $F(\psi)$.

Let $H(\mathcal{R})$ be the Hilbert space of functions \vec{u} with components quadratical summable over \mathcal{R} . In $H(\mathcal{R})$ we define the inner product

$$(5.8) \quad (\vec{u}, \vec{v})_{\mathcal{R}} = \int_{\mathcal{R}} \vec{u} \cdot \vec{v} dx dy.$$

The boundary values of the elements of $H(\mathcal{R})$ along \mathcal{B} form a second Hilbert space $H(\mathcal{B})$. In $H(\mathcal{B})$ we define

$$(5.9) \quad (\vec{u}, \vec{v})_{\mathcal{B}} = \int_{\mathcal{B}} \vec{u} \cdot \vec{v} d\mathcal{B}.$$

Finally we define the operators

$$(5.10) \quad \begin{cases} L^* = -(\partial_l t_{ik} a_{jk}^1) + t_{ik} c_{kj}, \\ B = (t_{ik} a_{kj}^1 n_l), \vec{n} \text{ is the outer-normal along } \mathcal{B}, \\ M^* = (B + M)^T, \\ R = -(\partial_l (t_{ik} a_{kj}^1)) + 2t_{ik} c_{kj}. \end{cases}$$

Applying Green's theorem we transform the inner-product $(L\vec{u}, \vec{v})_{\mathcal{R}}$ to

$$\begin{aligned}
(\vec{L}\vec{u}, \vec{v})_{\mathcal{R}} &= \int_{\mathcal{R}} v_i t_{ik} a_{kj}^{\perp} \partial_l u_j \, dx dy + \int_{\mathcal{R}} v_i t_{ik} c_{kj} u_j \, dx dy \\
&= \int_{\mathcal{B}} v_i t_{ik} a_{kj}^{\perp} n_l u_j \, d\mathcal{B} \\
&\quad - \int_{\mathcal{R}} u_j \partial_l t_{ik} a_{kj}^{\perp} v_i \, dx dy + \int_{\mathcal{R}} v_i t_{ik} c_{kj} u_j \, dx dy.
\end{aligned}$$

In terms of $L^{\vec{}}$, M and $M^{\vec{}}$ we obtain the fundamental relation

$$(5.11) \quad (\vec{L}\vec{u}, \vec{v})_{\mathcal{R}} - (\vec{u}, L^{\vec{}}\vec{v})_{\mathcal{R}} = (M^{\vec{}}\vec{v}, \vec{u})_{\mathcal{B}} - (\vec{v}, M\vec{u})_{\mathcal{B}}.$$

This relation is the startingpoint for existence-proofs and will be used in section 6.

For uniqueness-proofs we assume that $(t_{ik} a_{kj}^{\perp} \partial_l)$ is a symmetric operator and we find from (5.10) and (5.11):

$$(5.12) \quad F(\psi) = (\vec{L}\vec{u}, \vec{u})_{\mathcal{R}} = \frac{1}{2} (\vec{u}, B\vec{u})_{\mathcal{B}} + \frac{1}{2} (\vec{u}, R\vec{u})_{\mathcal{R}}.$$

We will require that the matrices B and R are both semi-definite (in the same sense), but not zero at the same time if $u \neq 0$). Since B and R are ordinary matrices this is a purely algebraic problem. Let (t_{ik}) be given by

$$(t_{ik}) = \begin{pmatrix} a & a_1 & a_2 & a_3 \\ b & b_1 & b_2 & b_3 \\ c & c_1 & c_2 & c_3 \end{pmatrix}.$$

The symmetry of $(t_{ik} a_{kj}^{\perp} \partial_l)$ implies

$$(5.13) \quad a_1 = -Kb; \quad a_2 = -Kc; \quad b_1 = a; \quad b_3 = -c; \quad a_3 = -b_2 = -c_1.$$

For the matrices B and R we find

$$(5.14) \quad B = \begin{pmatrix} aKn_1 - bKn_2 & bKn_1 + an_2 & cKn_1 + c_1n_2 \\ bKn_1 + an_2 & -an_1 + bn_2 & -c_1n_1 + cn_2 \\ cKn_1 + c_1n_2 & -c_1n_1 + cn_2 & -c_2n_1 - c_3n_2 \end{pmatrix},$$

$$(5.15) \quad R = \begin{pmatrix} (aK)_x - (bK)_y + 2cK & (bK)_x + a_y + 2c_1 & (cK)_x + c_{1y} \\ (bK)_x + a_y - 2c_1 & -a_x + b_y + 2c & -c_{1x} + c_y \\ (cK)_x + c_{1y} - 2c_2 & -c_{1x} + c_y - 2c_3 & -c_{2x} - c_{3y} \end{pmatrix}.$$

There are six functions to make B and R definite on \mathcal{N}_M .

If we choose $c = c_1 = c_2 = 0$, one can verify that $F(\psi)$ is the same as the Morawetz function (5.12). $c_1 = c_2 = 0$ leads to the case of Protter.

We consider the case of Morawetz, i.e. $c = c_1 = c_2 = 0$.

On that part of the boundary where no conditions are prescribed the matrix

$$(5.16) \quad B = \begin{pmatrix} \alpha K & \beta & 0 \\ \beta & -\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\alpha = an_1 - bn_2$, $\beta = bKn_1 + an_2$, must be semi-definite.

This means that the eigenvalues of B must be semi-definite, which is the case if and only if

$$(5.17) \quad \begin{cases} \det \begin{pmatrix} \alpha K & \beta \\ \beta & -\alpha \end{pmatrix} = -\alpha^2 K - \beta^2 \geq 0, \\ \text{trace} \begin{pmatrix} \alpha K & \beta \\ \beta & -\alpha \end{pmatrix} = \alpha(K-1) \text{ is semi-definite.} \end{cases}$$

Next we consider that part of the boundary where conditions of the type

$$(5.18) \quad p(x,y)\psi_x + q(x,y)\psi_y = 0$$

are prescribed. An equivalent formulation of these boundary conditions is

$$(5.18') \quad M\vec{u} = \begin{pmatrix} \lambda p & \lambda q & 0 \\ \mu p & \mu q & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{u} = 0,$$

where λ and μ are functions defined along the boundary.

We require that M is symmetric, hence

$$(5.18'') \quad M\vec{u} = \lambda \begin{pmatrix} p & q & 0 \\ q & q^2/p & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{u} = 0.$$

Now we must make $\vec{u} \cdot B\vec{u} = \vec{u} \cdot (B + M)\vec{u}$ semi-definite.

For $(B + M)$ we have

$$(5.19) \quad B + M = \begin{pmatrix} \alpha K + \lambda p & \beta + \lambda q & 0 \\ \beta + \lambda q & -\alpha + \lambda q^2/p & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Because of the symmetry of the matrix $(B + M)$ a necessary and sufficient condition is the semi-definiteness of the eigenvalues of $(B + M)$.

Analogous to (5.7) we find that

$$(5.20) \quad \begin{cases} \det \begin{pmatrix} \alpha K + \lambda p & \beta + \lambda q \\ \beta + \lambda q & -\alpha + \lambda q^2/p \end{pmatrix} \geq 0, \\ \text{trace} \begin{pmatrix} \alpha K + \lambda p & \beta + \lambda q \\ \beta + \lambda q & -\alpha + \lambda q^2/p \end{pmatrix} \text{ definite.} \end{cases}$$

We choose λ in such a way that the determinant vanishes, i.e.

$$(5.20a) \quad \lambda = \frac{-p(\alpha^2 K + \beta^2)}{\alpha p^2 + 2\beta pq - \alpha K q^2}.$$

Hence the second condition of (5.20) implies that

$$(5.20b) \quad \alpha(K-1) - \frac{(p^2 + q^2)(\alpha^2 K + \beta^2)}{\alpha p^2 + 2\beta pq - \alpha K q^2} \text{ is semi-definite.}$$

If we call the boundary without conditions \mathcal{B}_1 , and if we define $\mathcal{B}_2 = \mathcal{B} - \mathcal{B}_1$, we obtain the following requirements along the boundary \mathcal{B} :

$$(5.17a) \quad -(a^2 + Kb^2)(Kn_1^2 + n_2^2) \geq 0 \quad \text{along } \mathcal{B}_1,$$

$$(5.17b) \quad (K-1)(an_1 - bn_2) \text{ is semi-definite along } \mathcal{B}_1,$$

$$(5.20b) \quad (K-1)(an_1 - bn_2) - \frac{(a^2 + Kb^2)(Kn_1^2 + n_2^2)}{a(n_1p^2 + 2n_2pq - n_1Kq^2) + b(-n_2p^2 + 2n_1Kpq + n_2Kq^2)}$$

is semi-definite along \mathcal{B}_2 .

To simplify the conditions on R we set

$$(5.21) \quad a = a(x), \quad b = b(y).$$

The eigenvalues of R are simply

$$\rho_1 = 0; \quad \rho_2 = -K(y)(a_x - b_y) + bK'; \quad \rho_3 = a_x - b_y.$$

Therefore the conditions over the domain \mathcal{R} are:

$$(5.22) \quad -K(a_x - b_y) + bK' \quad \text{and} \quad a_x - b_y \quad \text{are semi-definite in } \mathcal{R}.$$

For boundary conditions of the first kind we have

$$(5.23) \quad \psi = 0 \rightarrow \frac{\partial \psi}{\partial s} = \nabla \psi \cdot \vec{s} = n_2 \psi_x - n_1 \psi_y = 0,$$

where \vec{s} is the unitvector along \mathcal{B} . From (5.18) we see that $p = n_2$ and $q = -n_1$. Hence (5.20b) can be written as:

$$(5.20b') \quad (a^2 + b^2) \cdot \frac{Kn_1^2 + n_2^2}{an_1 + bn_2} \quad \text{semi-definite along } \mathcal{B}_2.$$

Now we apply these formulae to boundary value problems of the first kind for an elliptic, a hyperbolic and a mixed equation, leading to the Dirichlet-problem, the Goursat-problem and the Tricomi-problem respectively.

The Dirichlet-problem - In the subsonic region of transonic flow we have $K(y) > 0$ and $K'(y) > 0$. Now $K'(y)$ may be very small, therefore conditions (5.22) suggest

$$a_x - b_y = 0.$$

According to (5.21) we find

$$a = dx + d_1, \quad b = dy + d_2,$$

where d , d_1 and d_2 are constants.

Condition (5.17a) reveals that boundary conditions must be prescribed along the whole boundary \mathcal{B} , a familiar property of elliptic equations. There remains the condition that

$$(5.24) \quad \vec{r} \cdot \vec{n} \text{ is semi-definite along } \mathcal{B},$$

where \vec{r} is a vector with components $(a, b) = (dx + d_1, dy + d_2)$.

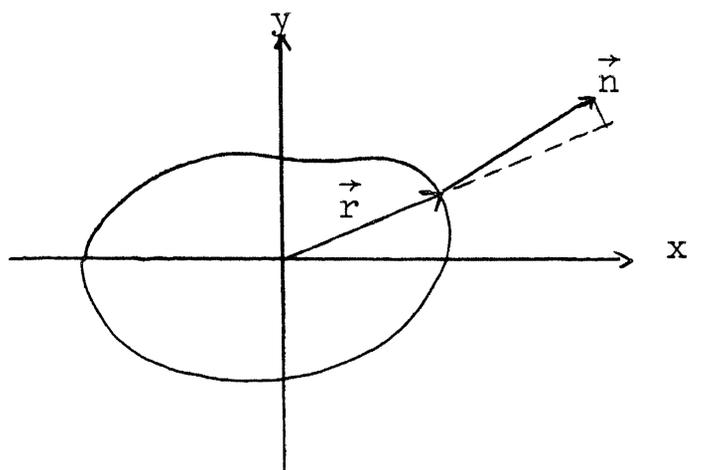


fig. 9

Figure 9 suggests

$$d = 1, d_1 = d_2 = 0,$$

for then condition (5.24) is satisfied by a wide class of boundary value problems.

The Goursat-problem - Along the characteristics we have

$$Kn_1^2 + n_2^2,$$

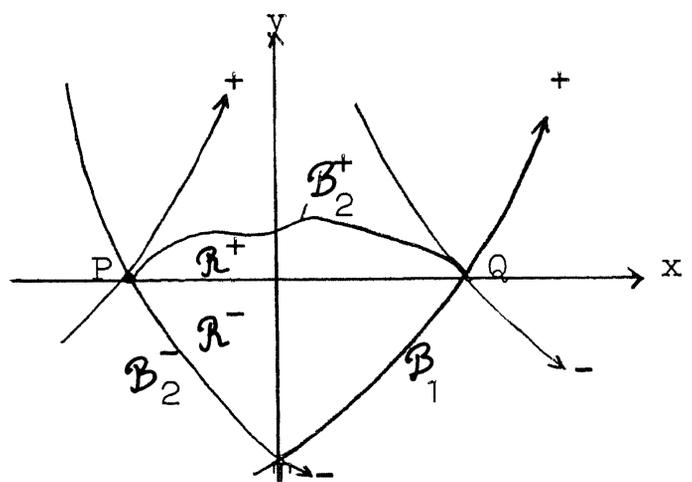


fig. 10

thus the conditions (5.17a) and (5.20b') are satisfied (see figure 10) along \mathcal{B}_1 and \mathcal{B}_2^-

Condition (5.22) is satisfied by every constant a and b . We take b positive and we define

$$\vec{r} = (a, b), \quad \vec{s} = (-a, b).$$

The following requirements are to be imposed on the boundary

- (a) $\vec{s} \cdot \vec{n} \geq 0$ along \mathcal{B}_1 ,
- (b) $\vec{r} \cdot \vec{n} \geq 0$ along \mathcal{B}_2^+ ,
- (c) $Kn_1^2 + n_2^2 \geq 0$ along \mathcal{B}_2^+ .

Because of the positivity of b , (a) is satisfied if $\vec{n}_T^+ \cdot \vec{s} \geq 0$ (see figure 11), or equivalently $\vec{n}_T^- \cdot \vec{r} \geq 0$. In other words: the slope along \mathcal{B}^- is everywhere greater than the slope of \vec{r} . In the same way we

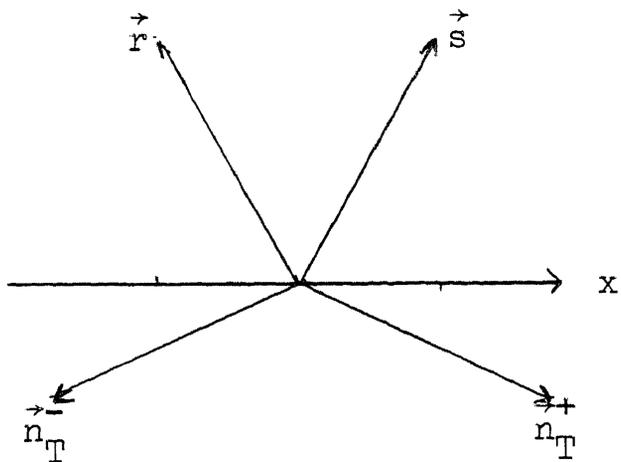


fig. 11

prove that if (b) is satisfied, the slope along \mathcal{B}^+ must be less than the slope of \vec{r} .

Combining these results we find that the maximal slope of \mathcal{B}_2^+ can not exceed the minimal slope of \mathcal{B}^- .

Condition (c) states that the boundary part \mathcal{B}_2^+ can not cut the same characteristic twice.

The Tricomi-problem - One may give the same proof as given for the

Goursat-problem. The only difference between the two problems is the sign of $K(y)$ along \mathcal{B}_2^+ . We obtain the same conditions (a) and (b) by choosing a and b to be constants with b positive.

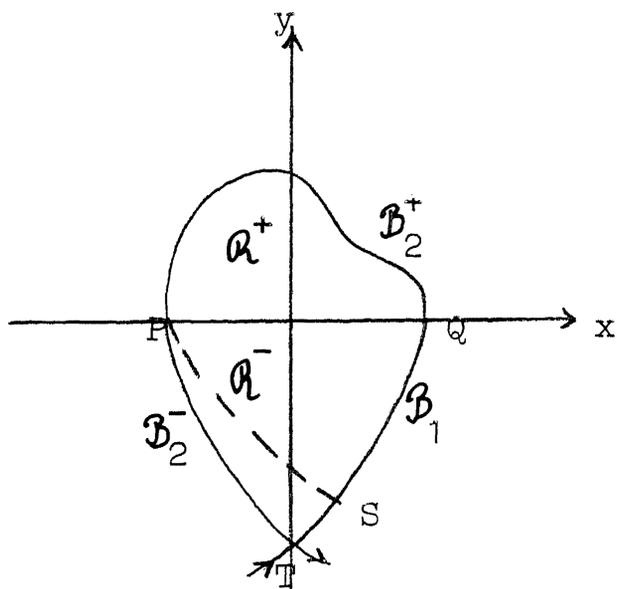


fig. 12

Condition (c) is always satisfied, so that the maximal slope of \mathcal{B}_2^+ must be less than or equal to the minimal slope of \mathcal{B}^- . We remark that it is not necessary for \mathcal{B}_2^- to be a characteristic. The conditions (a) and (b) imply the condition (5.20b') along \mathcal{B}_2^- , if this part of the

boundary lies within the characteristic triangle PQT and does not cut the same characteristic twice. Such a boundary value problem is called the Frankl'-problem.

Naturally one may soften the conditions on the elliptic boundary part by choosing non-constants a and b or by introducing functions c , c_1 and c_2 , as is done by Morawetz [25] and Protter [31, 33].

6. Weak and Strong solutions

We conclude this paper on the analytical aspects of the Tricomi-problem with some theorems concerning weak and strong solutions.

We call the solution u of equation (5.6')

$$(6.1) \quad L\vec{u} = \vec{f},$$

which satisfies the boundary condition

$$(6.2) \quad M\vec{u} = \vec{0},$$

a weak solution, if for all $\vec{v} \in \mathcal{N}_M^*$,

$$(6.3) \quad (\vec{v}, \vec{f})_{\mathcal{R}} \equiv (L^* \vec{v}, \vec{u})_{\mathcal{R}},$$

where M^* and L^* are defined in section 5 by (5.10).

Theorem I. An ordinary solution of (6.1) and (6.2) is a weak solution.

The proof follows immediately from (5.11) and the definition of a weak solution.

Theorem II. If the matrices B and R are definite in the opposite sense for all $\vec{v} \in \mathcal{N}_M^*$, then there exists a unique weak solution of (6.1) and (6.2).

Proof. Let \vec{v} be an element $\in \mathcal{N}_M^*$ and define the element \vec{g} by

$$(6.4) \quad \vec{g} = L^* \vec{v}.$$

In analogy to (5.12), we deduce that

$$(6.5) \quad (\vec{v}, L^* \vec{v})_{\mathcal{R}} = F^*(\psi) = -\frac{1}{2}(\vec{v}, B\vec{v})_{\mathcal{B}} + \frac{1}{2}(\vec{v}, R\vec{v})_{\mathcal{R}}.$$

From the condition of the theorem we have that equation (6.4) is uniquely solvable in \mathcal{N}_M^* , thus we can define the functional

$$(6.6) \quad \psi(L^* \vec{v}) = (\vec{v}, \vec{f})_{\mathcal{R}}, \quad \vec{v} \in \mathcal{N}_M^*$$

uniquely.

This functional $\Psi(L^{\leftrightarrow} \vec{v})$ is bounded, for according to Cauchy-Schwartz we have

$$|\Psi(L^{\leftrightarrow} \vec{v})| \leq \|\vec{v}\|_{\mathcal{R}} \cdot \|\vec{f}\|_{\mathcal{R}},$$

and from (6.5) we have

$$\|\vec{v}\|_{\mathcal{R}} \cdot \|L^{\leftrightarrow} \vec{v}\|_{\mathcal{R}} \geq \frac{1}{2} |(v, R\vec{v})_{\mathcal{R}}|,$$

thus

$$(6.7) \quad |\Psi(L^{\leftrightarrow} \vec{v})| \leq 2 \frac{\|\vec{v}\|_{\mathcal{R}}}{|(v, R\vec{v})_{\mathcal{R}}|} \cdot \|\vec{f}\|_{\mathcal{R}} \cdot \|L^{\leftrightarrow} \vec{v}\|_{\mathcal{R}}.$$

Now R is a symmetrical matrix, hence $\|\Psi\|$ is bounded by $2\|\vec{f}\|_{\mathcal{R}}/\|R\|$. The functional Ψ is a linear bounded functional defined on the range of L^{\leftrightarrow} . According to Hahn-Banach we can extend Ψ with the same norm over the whole space $H(\mathcal{R})$. If we now apply the representation-theorem of Riesz, we can write

$$(6.8) \quad \phi(L^{\leftrightarrow} \vec{v}) = (L^{\leftrightarrow} \vec{v}, \vec{u})_{\mathcal{R}},$$

for just one element \vec{u} .

From (6.6) and (6.8) we see that the equations (6.1) and (6.2) are uniquely solvable in the weak sense, proving the theorem.

The unique existence of a weak solution for the Tricomi problem was first proved by Morawetz [26].

We remark that the method described in section 5 may be used to prove the uniqueness and existence of weak solutions, where for ordinary solutions the method yields only uniqueness.

Besides the notion of weak solutions one has introduced strong solutions. A solution is called strong if it lies in the closure of the graph of (6.1) and (6.2). The theory concerning strong solutions is essentially developed by Friedrichs [5], Lax [23] and Lax and Phillips [24].

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